Standard Filtrations

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Def $M \in \mathcal{O}$ is said to have a standard filtration if there is a seq of submods

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M \quad \text{s.t.} \quad M^i = M_i / M_{i-1} \simeq \text{a Verma mod}$$

- n is said to be the filtration length. (Well-defined) ⟵ Useful
- $(M : M(\lambda))$ multiplication of $M(\lambda)$; ($[M : L(\lambda)]$ mult in Jordan-Hölder series)
- In general, M_i are NOT unique. ($M_1 \oplus M_2$), but multi and length are unique.

Prop. Let $M \in \mathcal{O}$ have a standard filtration.

1. If λ is maximal among the weights of M, then M has a submod isomorphic to $M(\lambda)$, and $M/M(\lambda)$ has a standard filtration.
2. If $M = M' \oplus M''$, then M' & M'' have standard filtrations.
3. M is free as a $\mathcal{U}(\mathfrak{n}^-)$-mod.

Pf. 1. Since $M_\lambda \neq 0$, take any nonzero $m_\lambda \in M_\lambda$. Then there exists a homo $\varphi : M(\lambda) \to M$ by $\varphi(v_\lambda) = m_\lambda$. Claim: φ is injective.

Consider a std fil $\quad 0 = M_0 \subset \cdots \subset \underset{\not\supseteq \varphi(M(\lambda))}{M_{i-1}} \subset \underset{\supseteq \varphi(M(\lambda))}{M_i} \subset M_{i+1} \cdots \subset M_n = M$

Then $\pi \circ \varphi : M(\lambda) \xrightarrow{\varphi} M_i \xrightarrow{\pi} M_i / M_{i-1} \simeq M(\mu)$. $\Rightarrow \mu \geq \lambda$.

By the maximality of λ, $\mu = \lambda$. Then $\pi \circ \varphi$ is an isom. $\Rightarrow \varphi$ injective.

$$0 = M_0 \subset \cdots \subset M_{i-1} = M_i / \varphi(M(\lambda)) \subset M_{i+1} / \varphi(M(\lambda)) \subset \cdots \subset M_n / \varphi(M(\lambda)) \cong M / M(\lambda).$$

is a std fil of $M/M(\lambda)$.

2. Induction on the length of the filtration. If length $= 0$ or 1, it is obvious.
Let λ be a maximal weight of M. Say $M'_\lambda \neq 0$.
Then $M(\lambda) \to M' \hookrightarrow M$ gives $M(\lambda) \hookrightarrow M'$ by (1).
Thus, $M/M(\lambda) \simeq M'/M(\lambda) \oplus M''$. By induction hypo, M'', $M'/M(\lambda)$ have std fil, so does M'.

3. Induction on the length. If length $= 0$ or 1, it is obv.
Let λ be a max weight. Then $0 \to M(\lambda) \to M \to M/M(\lambda) \to 0$

$M(\lambda)$, $M/M(\lambda)$ are $\mathcal{U}(\mathfrak{n}^-)$-free $\overset{?}{\Rightarrow}$ M $\mathcal{U}(\mathfrak{n}^-)$-free

Exercise (a) If M has a std fil and $\varphi : M \twoheadrightarrow M(\lambda)$, then $\ker\varphi$ has a std fil.

(b) $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, $\varphi : M(\lambda) \hookrightarrow M$, M has a std fil $\not\Rightarrow$ $\operatorname{coker}\varphi$ has a std fil.

Pf. (a) Step 1. Normalized the std fil:

Assume that $0 = M_0 \subset M_1 \cdots \subset M_n = M$, with $M_i/M_{i-1} \cong M(\lambda_i)$

Indeed we can take $\lambda_i \not< \lambda_j$ $\forall i<j$ by thm 3.1, (if $\lambda \not< \mu$, $\operatorname{Ext}(M(\lambda), M(\mu)) = 0$)

$0 \to M_i/M_{i-1} \to M_{i+1}/M_{i-1} \to M_{i+1}/M_i \to 0$ is a SES.

$\Rightarrow 0 \to M(\lambda_i) \to M_{i+1}/M_{i-1} \to M(\lambda_{i+1}) \to 0$ is a SES.

If $\lambda_i > \lambda_{i+1}$, $M_{i+1}/M_{i-1} = M(\lambda_i) \oplus M(\lambda_{i+1})$. Thus, we can rearrange the fil.

Let $0 = M_0 \subset M_1 \subset \cdots \subset \overset{\ker\varphi}{\overset{\cup}{M_k}} \subset \overset{\ker\varphi}{\overset{\not\subset}{M_{k+1}}} \subset \cdots \subset M_n = M$ be normalised.

Goal: $0 \subset M_0 \subset \cdots \subset M_k = (M_{k+1} \cap \ker\varphi) \subset (M_{k+2} \cap \ker\varphi) \subset \cdots \subset (M_n \cap \ker\varphi) = \ker\varphi$ is std

Step 2: $M_k = M_{k+1} \cap \ker\varphi$.

Then $\psi : M(\lambda_{k+1}) \simeq M_{k+1}/M_k \hookrightarrow M/M_k \xrightarrow{\varphi'} M/\ker\varphi \simeq M(\lambda)$ nonzero homo.

$\Rightarrow \lambda_{k+1} \le \lambda$. Claim $\lambda_{k+1} = \lambda$.

(If $\lambda_{k+1} < \lambda$, then there must exist $j > k+1$ st. $\lambda_j = \lambda$ (consider $(M : M(\lambda))$), which conflicts our assumption.)

Thus ψ is an isom. and $\ker\psi = \dfrac{\ker\varphi \cap M_{k+1}}{M_k} = \dfrac{\ker\varphi \cap M_{k+1}}{\ker\varphi \cap M_k} = 0$.

Step 3: $\dfrac{M_{i+1} \cap \ker\varphi}{M_i \cap \ker\varphi} \simeq M(\lambda_{i+1})$, $\forall i > k$

To simplicity, we consider a new std fil by mod out of M_k:

$$0 = V_k \subset V_{k+1} \subset \cdots \subset V_n = V, \text{ where } V_i = M_i/M_k. \quad V = M/M_k$$

Note that $\varphi' : V \twoheadrightarrow M(\lambda)$, with $\ker\varphi' = \ker\varphi/M_k$ and $V_{k+1} \underset{\varphi'}{\xrightarrow{\sim}} M(\lambda)$

Consider restriction of φ' on each V_i, denoted by φ'_i:

$$\begin{tikzcd} V_{k+1} \arrow[r, hook] \arrow[dr, "\varphi'_{k+1}"', "\sim"] & V_i \arrow[d, "\varphi'_i"] \\ & M(\lambda) \end{tikzcd}$$

$\Rightarrow V_{k+1} + \ker\varphi'_i = V_i$ $\quad S/S\cap T \cong S+T/T$

$$\frac{\ker\varphi \cap M_{i+1}}{\ker\varphi \cap M_i} = \frac{\overbrace{\ker\varphi' \cap V_{i+1}}^{\ker\varphi'_{i+1}}}{\ker\varphi' \cap V_i} \cong \frac{\ker\varphi' \cap V_{i+1} + V_i}{V_i} = \frac{V_{i+1}}{V_i} \cong M(\lambda_{i+1})$$

(b) $0 \to M(-3) \hookrightarrow M(1) \to L(1) \to 0$

Thm. If M has a std fil, then for all $\lambda \in \mathfrak{h}^*$, we have

$$(M : M(\lambda)) = \dim \operatorname{Hom}(M, M(\lambda)^\vee)$$

Pf. Induction on the length of M: if $M = M(\mu)$, $\dim \operatorname{Hom}(M(\mu), M(\lambda)^\vee) = \delta_{\mu\lambda}$ by thm 3.3

Suppose $0 \to N \to M \to M(\mu) \to 0$. There is a LES by Ext functor.

$$0 \to \underbrace{\operatorname{Hom}(M(\mu), M(\lambda)^\vee)}_{\dim = \delta_{\mu\lambda}} \to \operatorname{Hom}(M, M(\lambda)^\vee) \to \underbrace{\operatorname{Hom}(N, M(\lambda)^\vee)}_{\dim = (N, M(\lambda))} \to \underbrace{\operatorname{Ext}(M(\mu), M(\lambda)^\vee)}_{=0} \to \cdots$$

$\Rightarrow \dim \operatorname{Hom}(M, M(\lambda)^\vee) = (N, M(\lambda)) + \delta_{\mu\lambda} = (M, M(\lambda))$

Projectives in $\mathcal{O}$

Def. P is projective: $\forall M \twoheadrightarrow N$ and $P \to N$, $\exists P \to M$ st.

$$\begin{tikzcd} & P \arrow[d] \arrow[dl] \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

Q is injective: $\forall N \hookrightarrow M$ and $N \to Q$, $\exists M \to Q$ st.

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[dl] \\ & Q \end{tikzcd}$$

enough projectives: $\forall M \in \mathcal{O}$, $\exists$ proj $P \in \mathcal{O}$ and $P \twoheadrightarrow M$.

injectives: $\forall M \in \mathcal{O}$, $\exists$ inj $Q \in \mathcal{O}$ and $M \hookrightarrow Q$

projective cover: $(P_M, \pi : P_M \twoheadrightarrow M)$ with

$$\begin{tikzcd} & P \arrow[d, "\pi"] \arrow[dl, dashed] \\ N \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Prop. $\mathcal{O}$ contains some projectives:

a) Suppose $\lambda \in \mathfrak{h}^*$ dominant. Then $M(\lambda)$ is projective in $\mathcal{O}$.

b) If $P \in \mathcal{O}$ is projective, while $\dim L < \infty$, then $P \otimes L$ is projective.

Pf. a)
$$\begin{tikzcd} M \arrow[r, "\pi"] & N \ni v = \varphi(v^+) \arrow[r] & 0 \\ & M(\lambda) \ni v^+ \arrow[u, "\varphi"] \arrow[ul, dashed] \end{tikzcd}$$

Since $\operatorname{Im} \varphi \in N^\chi = \{v \in N : zv = \chi(z) \cdot v\}$, it suffices to consider M, N are in $\mathcal{O}^\chi$.

Since π surjective, $\exists$ nonzero $m \in M_\lambda$ st. $\pi(m) = v = \varphi(v^+)$.

Since λ is dominant, $\mathfrak{n}^+ \cdot m = 0$. By universal property of Verma, $\exists \varphi' : M(\lambda) \to M$, $v^+ \mapsto m$

b) Goal: $\operatorname{Hom}(P \otimes L, \bullet)$ exact.

$$\mathrm{Hom}(P\otimes L, M) \cong \mathrm{Hom}(P, \mathrm{Hom}(L,M)) \cong \mathrm{Hom}(P, L^*\otimes M)$$

By thm 1.1(d), $L^*\otimes\bullet$ is exact $\Rightarrow$ $\mathrm{Hom}(P\otimes L, \bullet)$ exact

Exercise. If Q injective, L fin dim, then $Q\otimes L$ injective

$$\mathrm{Hom}(\bullet, Q\otimes L) \cong \mathrm{Hom}(\bullet, \mathrm{Hom}(L,Q)) \cong \mathrm{Hom}(\bullet\otimes L, Q)$$

Thm. Category $\mathcal{O}$ has enough projectives.

pf. Step 1. $L(\lambda)$ has a projective obj mapped onto.

(Proj. $\twoheadrightarrow M(\lambda) \twoheadrightarrow L(\lambda)$)

Let $\mu=\lambda+n\rho$ be dominant for n sufficiently large.

$M(\mu)\otimes L(n\rho)$ is projective and has a std fil by thm 3.6 with quotient $M(\lambda_i)$, where λ_i runs over $\{\mu+\text{weight of } L(n\rho)\}$.

By rearranging the std fil, we can have $0\to M_{n-1}\to M(\mu)\otimes L(n\rho)\to M(\mu-n\rho)\to 0$

i.e. $M(\mu)\otimes L(n\rho)$ has a quotient isom to $M(\lambda)$ and $(M(\mu)\otimes L(n\rho): M(\lambda))=1$.

Step 2. Induction on the length of M.

If $\ell(M)=1$, step 1 has proved.

Assume $0\to L(\lambda)\to M\to N\to 0$. By induction hypo, $\exists$
$$\begin{tikzcd} & P \arrow[dl, "\hat\varphi"'] \arrow[d, "\varphi"] & \\ M \arrow[r, "\pi"] & N \arrow[r] & 0 \end{tikzcd}$$

① If $\hat\varphi$ is surj then it is done.

② If $\hat\varphi$ is not surj then $\hat\varphi(P)\cap L(\lambda)=0$. (Otherwise, $L(\lambda)\subset\hat\varphi(P)$ and $\hat\varphi$ surj)
$\hat\varphi(P) \supset \pi^{-1}(N)$

Consider $i: N\to M$ by $n\mapsto\hat\varphi\circ\varphi^{-1}(n)$

Well-defined: $\hat\varphi(\ker\varphi)\subset\ker\pi=L(\lambda)\Rightarrow\hat\varphi(\ker\varphi)=0$.

Morphism: $\forall x\in\mathfrak{g}$, $\hat\varphi(\varphi^{-1}(xn))=\hat\varphi(x\varphi^{-1}(n)+\ker\varphi)=x\hat\varphi(\varphi^{-1}(n))$

$\pi\circ i=id_N$: $\forall n\in N$, $\pi\circ i(n)=\pi\hat\varphi\circ\varphi^{-1}(n)=\varphi\circ\varphi^{-1}(n)=n$.

Thus, $M=N\oplus L(\lambda)$.

Then the direct sum of projectives of N & $L(\lambda)$ is a projectives of M.